A class of Lorenz-type systems, their factorizations and extensions

I. Kunin[1] and A. Runov[2]

February 1, 2008

Abstract

It is well known that the Lorenz system has Z_2-symmetry. Using introducted in [3] topological covering-coloring a new representation for the Lorenz system is obtained. Deleting coloring leads to the factorized Lorenz system that is in a sense more fundamental than the original one. Finally, Z_n extensions define a class of Lorenz-type systems. The approach admits a natural generalization for regular and chaotic systems with arbitrary symmetries.

There are more than a thousand publications related to the Lorenz system and the number is increasing. Some explanations for this popularity are given, e.g. in [1, 2]. Here the Lorenz system is realized in a non-traditional form that may motivate a new approach to investigating chaotic systems with symmetries. The idea of this approach has been demonstrated in [3] for the simple and instructive example of the Duffing oscillator with Z_2-symmetry. The basic components are: realization of the symmetry by topological covering-coloring, factorization by deleting the coloring, and the following Z_n-extension. As a result a class of Lorenz-type systems is defined. An open and intriguing problem: comparison of chaos for elements of the class.

[1] Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA, e-mail kunin@uh.edu

[2] Department of Theoretical Physics, St. Petersburg University, Uljanovskaja 1, St. Petersburg, Petrodvorez, 198504, Russia

Let us start with the standard form of the Lorenz system

$$\begin{cases} \dot{X} = \sigma(Y - X) \\ \dot{Y} = (r - Z)X - Y \\ \dot{Z} = -bZ + XY \end{cases} \tag{1}$$

and the corresponding canonical strange attractor (Fig. 1). It is convenient to use the following normalized form L_2

$$\begin{cases} \dot{x} = y \\ \dot{y} = ((1-\gamma)x^2 - 1 + z)x - \mu y \\ \dot{z} = \beta(\gamma x^2 - z) \end{cases} \tag{2}$$

related to (1) by (for $r > 1$)

$$\begin{aligned} t &= \sqrt{(r-1)\sigma}\, t_L \\ x &= \frac{1}{\sqrt{(r-1)b}} X \\ y &= \frac{1}{r-1}\sqrt{\frac{\sigma}{b}}(Y - X) \\ z &= \frac{1}{r-1}\left(Z - \frac{1}{2\sigma}X^2\right) \\ \mu &= \frac{1+\sigma}{\sqrt{(r-1)\sigma}}, \quad \beta = \frac{b}{\sqrt{(r-1)\sigma}}, \quad \gamma = 1 - \frac{b}{2\sigma} \end{aligned} \tag{3}$$

The system L_2 has 3 fixed points: $(0,0,0)$, $(\pm 1, 0, \gamma)$ and Z_2-symmetry: $(x,y) \to (-x,-y)$. Corresponding attractor is shown on Fig. 2. Here we have an analogy with the Duffing system [3] motivating the similar transformation of topological covering

$$\begin{aligned} x_1 &= x_1(x,y) = \frac{x^2 - y^2}{r}, \\ y_1 &= y_1(x,y) = \frac{2xy}{r}. \end{aligned} \qquad r = \sqrt{x^2 + y^2} \tag{4}$$

The corresponding colored trajectories are shown in Figs. 3, 4.

The meaning of the transformation becomes more transparent in polar coordinates: it is $(r,\phi) \to (r, 2\phi)$. The symmetry of the system L_2 is equivalent to $f(r,\phi) = f(r, \phi + \pi)$, where f is an arbitrary function on the states of L_2. Correspondingly, in new coordinates $f(r,\phi) = f(r, \phi + 2\pi)$, i. e. one

deals with two identical planes marked by colors. The equations for this representation of the original Lorenz system are uniquely defined by the indicated above transformations. Their explicit form can be easily obtained but is far from being elegant (really it is cumbersome) and is not given here.

The erasing of the colors is equivalent to gluing together two underlying planes, i. e. defining a new, more simple and fundamental factorized Lorenz system L_1 (Fig. 5). As above the explicit equations can be written, but they are cumbersome, and we do not present them here. The system L_1 carries all essential information about L_2 except the symmetry. Thus the Lorenz system may be naturally called a Z_2-extension of the L_1 system. This motivates to define L_n as Z_n-extension of L_1, i. e. the similar transformation to the corresponding system with Z_n-symmetry. The example for L_3 is given in Fig. 6. Thus we have defined the class of Lorenz-type systems. Any two elements of the class are related by compositions of appropriate factorizations and extensions. It is natural to call these elements a class of Z-equivalent systems (with a canonical representative L_1).

The approach admits a generalization to different families of discrete and continuous symmetries as well as wider classes of transformations. The scheme of such generalizatins will be given elsewhere.

References

[1] C. Sparrow, The Lorenz equations: bifurcations, chaos, and strange attractors. Springer-Verlag, N. Y., 1982.

[2] E. A. Jackson. Perspective of nonlinear dynamics. Cambridge Univ. Press, N. Y., vol. 2, 1990.

[3] I. Kunin and A. Runov, e-print math.DS/0105147.

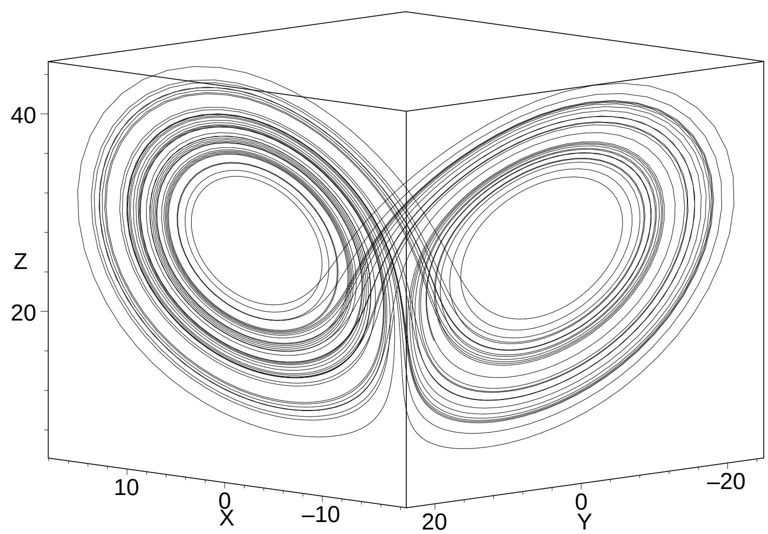

Figure 1: Standard Lorenz system.

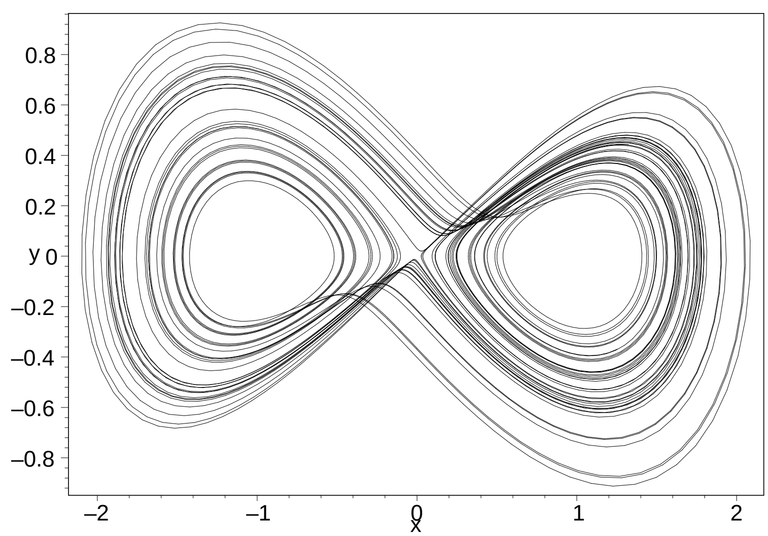

Figure 2: Rescaled Lorenz system (L_2).

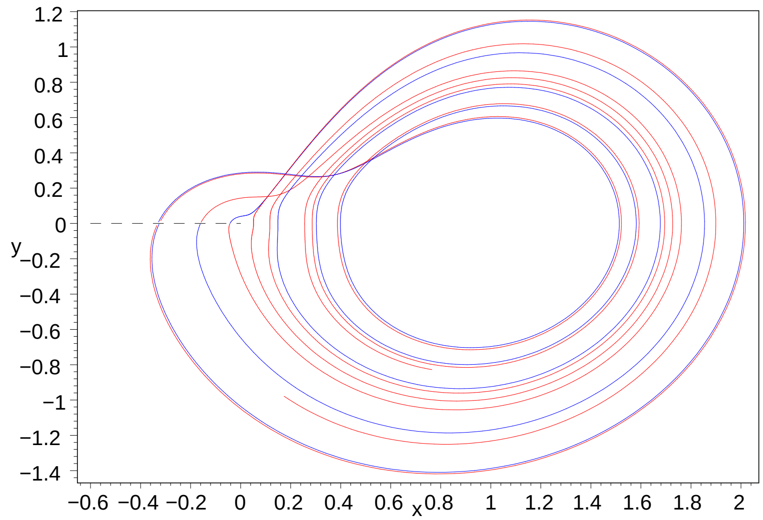

Figure 3: Covered (colored) L_2 system, x_1 — y_1 projection, (short piece of trajectory).

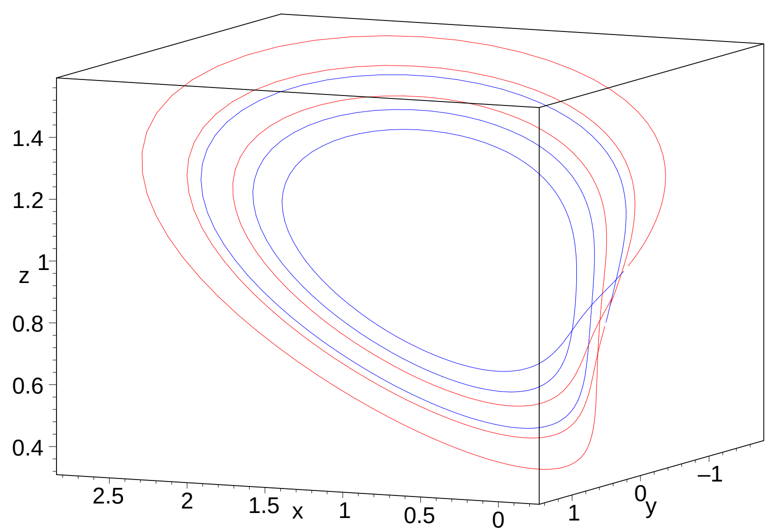

Figure 4: Covered (colored) L_2 system, 3-D projection,(short piece of trajectory).

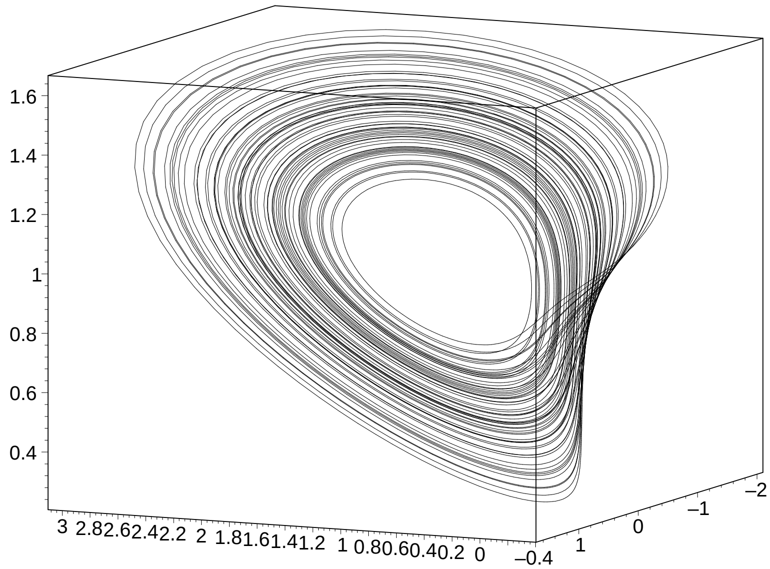

Figure 5: Factorized Lorenz attractor (L_1).

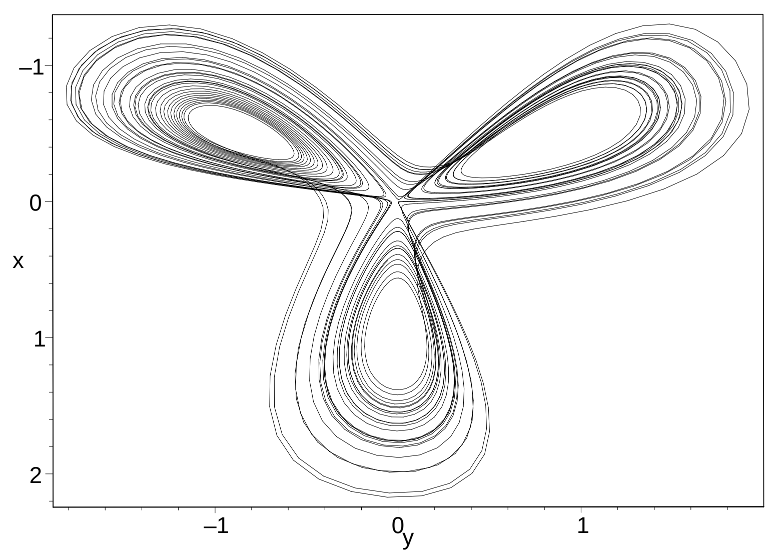

Figure 6: Attractor of L_3.